Summary Chapter 2

Basic vector operations Need to know

$$grad(p) = \overline{\nabla} p = \begin{pmatrix} \partial \bullet / \partial x \\ \partial \bullet / \partial y \\ \partial \bullet / \partial z \end{pmatrix} p = \begin{pmatrix} \partial p / \partial x \\ \partial p / \partial y \\ \partial p / \partial z \end{pmatrix}$$

$$div(\overline{V}) = \overline{\nabla} \cdot \overline{V} = \begin{pmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$curl(\overline{V}) = \overline{\nabla} \times \overline{V} = \begin{pmatrix} \partial / \partial x \\ \partial / \partial y \\ \partial / \partial z \end{pmatrix} \times \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \partial V_z / \partial y - \partial V_y / \partial z \\ \partial V_x / \partial z - \partial V_z / \partial x \\ \partial V_y / \partial x - \partial V_x / \partial y \end{pmatrix}$$

Gradient of a scalar You only need to know the formula for Cartesian coordinates

$$\overline{\nabla} p = \frac{\partial p}{\partial x}\overline{i} + \frac{\partial p}{\partial y}\overline{j} + \frac{\partial p}{\partial z}\overline{k}$$

$$\overline{\nabla} p = \frac{\partial p}{\partial r}\overline{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\overline{e}_\theta + \frac{\partial p}{\partial z}\overline{e}_z$$

$$\overline{\nabla} p = \frac{\partial p}{\partial r}\overline{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\overline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi}\overline{e}_\varphi$$

Directional derivative of a scalar Need to know

$$\frac{dp}{ds} = \frac{\partial p}{\partial x}\frac{dx}{ds} + \frac{\partial p}{\partial y}\frac{dy}{ds} + \frac{\partial p}{\partial z}\frac{dz}{ds} = \overline{\nabla} p \cdot \frac{d\overline{r}}{ds} = \overline{\nabla} p \cdot \overline{n}_s$$

Divergence of a vector field You only need to know the formula for Cartesian coordinates

Cartesian: $\overline{\nabla} \cdot \overline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

Cylindrical: $\overline{\nabla} \cdot \overline{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$

Spherical: $\overline{\nabla} \cdot \overline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$

Stokes Theorem **Need to know**

$$\oint_C \overline{A}\cdot d\overline{s} \equiv \iint_S \left(\overline{\nabla}\times\overline{A}\right)\cdot d\overline{S}$$

Divergence Theorem **Need to know**

$$\oiint_S \overline{A}\cdot d\overline{S} \equiv \oiiint_V \left(\overline{\nabla}\cdot\overline{A}\right) d\mathcal{V}$$

Gradient Theorem **Need to know**

$$\oiint_S p\, d\overline{S} \equiv \oiiint_V \overline{\nabla} p\, d\mathcal{V}$$

Substantial derivative **Need to know**

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\overline{V}\cdot\overline{\nabla}\right)$$

Continuity equation **Need to know**

$$\frac{\partial}{\partial t}\iiint_V \rho d\mathcal{V} + \iint_S \rho\overline{V}\cdot d\overline{S} = 0$$ Integral form

$$\frac{\partial\rho}{\partial t} + \overline{\nabla}\cdot\rho\overline{V} = 0$$ Differential form

$$\frac{D\rho}{Dt} + \rho\overline{\nabla}\cdot\overline{V} = 0$$ Substantial derivative form

Momentum equation **You need to know what the terms in these models physically mean**

$$\frac{\partial}{\partial t}\iiint_V \rho\overline{V} d\mathcal{V} + \iint_S \left(\rho\overline{V}\cdot d\overline{S}\right)\overline{V} = -\iint_S p d\overline{S} + \iiint_V \overline{f}\rho d\mathcal{V} + \overline{F}_{\text{visc}}$$ Integral form

$$\frac{\partial\left(\rho\overline{V}\right)}{\partial t} + \left(\overline{\nabla}\cdot\rho\overline{V}\right)\overline{V} = -\overline{\nabla}p + \rho\overline{f} + \overline{\mathrm{F}}_{\text{visc}}$$ Differential form

$$\rho\frac{D\overline{V}}{Dt} = -\overline{\nabla}p + \rho\overline{f} + \overline{\mathrm{F}}_{\text{visc}}$$ Substantial derivative form

$$\frac{\partial}{\partial x}\left(\rho u^2\right)+\frac{\partial}{\partial y}\left(\rho uv\right)+\frac{\partial}{\partial z}\left(\rho uw\right)=-\frac{\partial p}{\partial x}$$
$$\frac{\partial}{\partial x}\left(\rho vu\right)+\frac{\partial}{\partial y}\left(\rho v^2\right)+\frac{\partial}{\partial z}\left(\rho vw\right)=-\frac{\partial p}{\partial y}$$
$$\frac{\partial}{\partial x}\left(\rho wu\right)+\frac{\partial}{\partial y}\left(\rho wv\right)+\frac{\partial}{\partial z}\left(\rho w^2\right)=-\frac{\partial p}{\partial z}$$

Euler Equation

Energy equation **You need to know what the terms in these models physically mean**

$$\iiint_V \rho q dV+\dot{Q}_{\text{visc}}-\oiint_S p d\overline{S}\cdot\overline{V}+\iiint_V \rho\overline{f}dV\cdot\overline{V}+\dot{W}_{\text{visc}}=$$
$$\frac{\partial}{\partial t}\iiint_V \rho\left(e+\frac{1}{2}\overline{V}\cdot\overline{V}\right)dV+\oiint_S\left(\rho\overline{V}\cdot d\overline{S}\right)\left(e+\frac{1}{2}\overline{V}\cdot\overline{V}\right)$$

Integral form

$$\frac{\partial}{\partial t}\left(\rho E\right)+\overline{\nabla}\cdot\rho E\overline{V}=\rho\dot{q}-\overline{\nabla}\cdot p\overline{V}+\rho\overline{f}\cdot\overline{V}+\dot{Q}'_{\text{visc}}+\dot{W}'_{\text{visc}}$$

Differential form

$$\rho\frac{D\left(e+V^2/2\right)}{Dt}=\rho\dot{q}-\overline{\nabla}\cdot\left(p\overline{V}\right)+\rho\left(\overline{f}\cdot\overline{V}\right)+\dot{Q}'_{\text{visc}}+\dot{W}'_{\text{visc}}$$

Substantial Derivative form

Streamline **Need to know**

$$\overline{V}\times d\overline{s}=\begin{pmatrix} vdz-wdy \\ wdx-udz \\ udy-vdx \end{pmatrix}=0$$

In 3D space

$$\frac{dy}{dx}=\frac{v}{u}$$

In 2D space

Vorticity **Need to know**

$$\overline{\xi}=2\overline{\omega}=\begin{pmatrix} \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \end{pmatrix}=\overline{\nabla}\times\overline{V}$$

Circulation **Need to know**

$$\Gamma=-\oint_C \overline{V}\cdot d\overline{s}$$

$$\Gamma=-\iint_S\left(\overline{\nabla}\times\overline{V}\right)\cdot d\overline{s}$$

Stream function Need to know

$$\rho u = \frac{\partial \psi}{\partial y} \qquad \rho V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$\rho v = -\frac{\partial \psi}{\partial x} \qquad \rho V_\theta = -\frac{\partial \psi}{\partial r}$$

Compressible form (Cartesian / Cilindrical)

$$u = \frac{\partial \psi}{\partial y} \qquad V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$v = -\frac{\partial \psi}{\partial x} \qquad V_\theta = -\frac{\partial \psi}{\partial r}$$

Incompressible form (Cartesian / Cilindrical)

Velocity components in Cartesian coordinates Need to know

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}.$$

Velocity components expressed in the potential in polar coordinates Need to know

$$u_r = \frac{\partial \phi}{\partial r}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$